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On Shortest Cocycle Covers of Graphs

FRANÇOIS JAEGER

LSD(IMAG), BP 68, 38402 St. Martin d'Hères Cedex, France

ABDELKADER KHELLADI

U.S.T.H.B., Institut de Mathématiques, BP 139 Bab Ezzouar, Algérie

AND

MICHEL MOLLARD

LSD(IMAG), BP 68, 38402 St. Martin d'Hères Cedex, France

A cocycle (resp. cycle) cover of a graph G is a family C of cocycles (resp. cycles) of G such that each edge of G belongs to at least one member of C. The length of C is the sum of the cardinalities of its members. While it is known (see [5, 6]) that every bridgeless graph $G=(V, E)$ has a cycle cover of length not greater than $\frac{5}{3}|E|$, it is shown that there exists no $\alpha<2$ such that every loopless graph $G=(V, E)$ has a cocycle cover with length not greater than $\alpha|E|$. To do this, cocycle covers of minimal length are determined for the complete graphs, thus solving a problem stated at the end of [6].

1. INTRODUCTION

The graphs we consider are finite, undirected, and have at least one edge. For definitions not given here, the reader should refer to [1, or 2].

Let $G=(V, E)$ be a graph and $S \subseteq V$ be a set of vertices. The *cocycle of G defined by S*, which we denote by $\omega_G(S)$, is the set of edges of G with exactly one end in S. A *cocycle of* G is any set of edges of the form $\omega_G(S)$ for some $S \subseteq V$. A *vertex-cocycle* of G is any set of edges of the form $\omega_G(\{v\})$ for some $v \in V$.

A *cycle* (resp. *cocycle*) *cover* of the graph G is a family C of cycles (resp. cocycles of G such that each edge of G belongs to at least one member of C. The *length* of C is the sum of the cardinalities of its members and is denoted by $l(C)$. Note that the graph G admits a cycle (resp. cocycle) cover if and only if it has no bridges (resp. loops).

Recently several authors have studied the length of cycle covers of

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graphs (see [3–6]). In particular it is shown independently in [5, 6] that every bridgeless graph $G=(V, E)$ has a cycle cover of length not greater than $\frac{5}{3}|E|$. In fact a wide extension of this result to regular matroids is proved in [6]. This leads us to consider similar problems for the cocycle matroids of graphs, that is, to study the length of cocycle covers of loopless graphs.

Clearly every loopless graph $G=(V, E)$ has a cocycle cover C with $l(C)<2|E|$ (use all vertex-cocycles but one). A first basic question is the following: does there exist $\alpha<2$ such that every loopless graph $G=(V, E)$ has a cocycle cover C with $l(C) \leqslant \alpha|E|$? We shall prove that the answer is negative. To do this, we shall determine the minimum length of a cocycle cover of the complete graph K_n on n vertices for every n, thus solving the problem stated at the end of [6]. It turns out that, for all $n \geqslant 2$, $n \neq 4$, $n \neq 8$, one can obtain a cocycle cover of minimum length of K_n by taking all vertex-cocycles except one. Our proof is based on a reformulation of the problem in the following terms: what is the minimum of the sum of mutual distances of n points in a k-dimensional cube $(k \geqslant n-1)$?

2. Geometric Formulation of the Problem

2.1. We recall that the k-cube $(k \geqslant 1)$ is the simple graph Q_k whose vertex-set is $[GF(2)]^k$ and where $x=(x_1, \ldots, x_k)$ and $y=(y_1, \ldots, y_k)$ are adjacent iff $x+y$ has exactly one non-zero component. If x and y are vertices of Q_k, the number of non-zero components of $x+y$ is the distance between x and y in Q_k and is denoted by $d(x, y)$ (thus $d(x, y)$ is the usual Hamming distance of x and y).

2.2. Let $n \geqslant 2$. We denote by V_n and E_n the vertex-set and edge-set of K_n, respectively. For distinct vertices u, v of K_n, the edge with ends u and v is denoted by $\{u, v\}$.

Let $\mathscr{S}=(S_1, \ldots, S_k)$ be a k-tuple $(k \geqslant 1)$ of subsets of V_n. We associate to $\mathscr{S}$ a mapping from V_n to $[GF(2)]^k$ which we denote by s and which we define as follows. For every v in V_n, $s(v)=(s_1(v), \ldots, s_k(v))$, where $s_i(v)=1$ if $v \in S_i$ and $s_i(v)=0$ otherwise $(i=1, \ldots, k)$. Clearly the correspondence $\mathscr{S} \rightarrow s$ is a bijection from the set of k-tuples of subsets of V_n to the set of mappings from V_n to $[GF(2)]^k$.

Now we shall need the following properties.

(i) *$C=(\omega_{K_n}(S_i), i=1, \ldots, k)$ is a cocycle cover of K_n if and only if the mapping s associated to $(S_1, \ldots, S_k)$ is injective.*

Indeed, for distinct vertices u, v of K_n and $i \in \{1, \ldots, k\}$, the edge $\{u, v\}$

belongs to $\omega_{K_n}(S_i)$ iff $s_i(u) \neq s_i(v)$. Hence $\{u,v\}$ belongs to some member of C iff $s(u) \neq s(v)$, and (i) follows immediately.

(ii) $\sum_{i=1}^{k} |\omega_{K_n}(S_i)| = \sum_{\{u,v\} \in E_n} d(s(u), s(v))$.

Indeed it is easy to see that both sides of this equation represent the number of pairs $(\{u, v\}, i)(\{u, v\} \in E_n,\ i \in \{1,..., k\})$ such that $\{u, v\} \in \omega_{K_n}(S_i)$, or equivalently $s_i(u) \neq s_i(v)$.

The following property can now easily be deduced from (i) and (ii).

(iii) *The minimum length of a cocycle cover of K_n consisting of exactly k cocycles is equal to the minimum of the quantity $\sum_{\{u,v\} \in E_n} d(s(u), s(v))$ taken over all injective mappings s from V_n to $[GF(2)]^k$.*

Remarks. —It follows from (i) that there exists a cocycle cover of K_n using exactly k cocycles iff $n \leqslant 2^k$.

—Let us distinguish a vertex v_0 of V_n. Every cocycle of K_n can be written uniquely in the form $\omega_{K_n}(S)$, where $S \subseteq V_n - \{v_0\}$. Hence every k-tuple of cocycles of K_n can be written uniquely in the form $(\omega_{K_n}(S_i),\ i = 1,..., k)$, where $S_i \subseteq V_n - \{v_0\}$ for $i = 1,..., k$. Then, by (i), the cocycle covers of K_n consisting of exactly k cocycles are in bijective correspondence with the injective mappings s from V_n to $[GF(2)]^k$ such that $s(v_0) = (0, 0,..., 0)$.

2.3. For every set A of vertices of Q_k, we denote by $D(A)$ the sum $\sum d(x, y)$, where each unordered pair of distinct vertices x, y of A appears once. We denote the minimum length of a cocycle cover of K_n by $l(K_n)$.

PROPOSITION 1. *For every $n \geqslant 2$, $k \geqslant n-1$:*

$$\min_{\substack{A \subseteq [GF(2)]^k \\ |A| = n}} D(A) = l(K_n)$$

Proof. First we note that a cocycle cover of K_n consisting of j cocycles with $j < k$ can be converted into a cocycle cover consisting of k cocycles, and with the same length, by simply adding $k - j$ empty cocycles. It then follows from **2.2**(iii) that the minimum length of a cocycle cover of K_n consisting of at most k cocycles is equal to $\min D(A)$,

$$A \subseteq [GF(2)]^k$$
$$|A| = n.$$

Then we observe that a cocycle cover of K_n with minimum length has at most $n-1$ non-empty elements. This is because every non-empty cocycle of

K_n has cardinality at least $n-1$, and there exists a cocycle cover consisting of $n-1$ cocycles of cardinality $n-1$ (all vertex-cocycles except one). Hence $l(K_n)$ is equal to the minimum length of a cocycle cover consisting of at most $n-1$ cocycles, and this clearly remains true if $n-1$ is replaced by some larger number. This completes the proof.

3. The Main Result

3.1. Let $n \geqslant 2$, $k \geqslant n-1$. A set A of n vertices of Q_k will be called *extremal* if $D(A) = l(K_n)$ (see Proposition 1). The *n-star* is the simple graph on n vertices which has one vertex of degree $n-1$, the other vertices being of degree 1.

Theorem 1. For every $n \geqslant 2$, $l(K_n) = (n-1)^2 - \varepsilon(n)$, where $\varepsilon(n) = 1$ if $n = 4$ or $n = 8$, $\varepsilon(n) = 0$ otherwise. Moreover, for every $k \geqslant n-1$, a set A of n vertices of Q_k is extremal if and only if the subgraph of Q_k induced by A is isomorphic to one of the following graphs:

—The n-star, for all $n \geqslant 2$, $n \neq 4$, $n \neq 8$.

—For $n = 4 + i$, $i = 0, \ldots, 4$, the graph G_n obtained from the 3-dimensional cube Q_3 by deleting $4 - i$ vertices belonging to a square (where, for $i = 2$, we delete two adjacent vertices). The graphs $G_4, \ldots, G_8$ are depicted on Fig. 1.

3.2. We recall some basic properties of the k-dimensional cube Q_k needed for the proof. First, Q_k is a bipartite graph, and we assume from

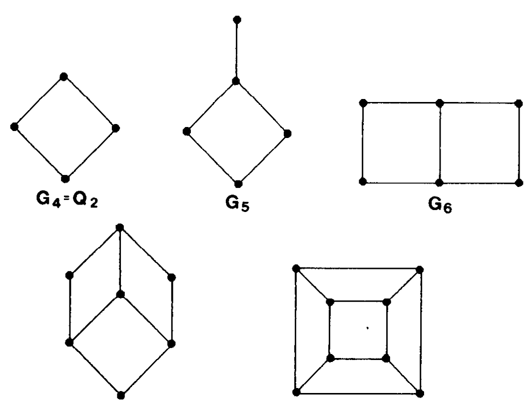

Figure 1

now on that we have defined a good coloring of its vertices with two colors we denote by 0 and 1. A second useful property is stated below.

LEMMA. If x and y are two vertices of Q_k with $d(x, y) = q$ $(1 \leqslant q \leqslant k)$, then there exist exactly q vertices z such that $d(z, x) = 1$ and $d(z, y) = q - 1$.

3.3. An induced subgraph G of Q_k is called *distance-preserving* if the usual distance-function defined by G on its vertex-set coincides with the restriction of the Hamming distance d to this set. It follows immediately from the biparticity of Q_k that all induced subgraphs of Q_k of diameter at most 3 are distance-preserving. In particular, the n-stars $(n \geqslant 2)$ and the graphs G_n $(n = 4,..., 8)$ occurring in the statement of the theorem are distance-preserving. It is then easy to compute $D(A)$ when the subgraph of Q_k induced by A is isomorphic to one of these graphs. We obtain $(n-1)^2$ for the n-star $(n \geqslant 2)$, and 8, 16, 25, 36, 48 for G_4, G_5, G_6, G_7, G_8, respectively. We conclude that:

(i) *If A is a subset of n vertices of Q_k which induces a subgraph isomorphic to an n-star $(n \geqslant 2,\ n \neq 4,\ n \neq 8)$ or to G_n $(n = 4,..., 8)$, then* $D(A) = (n-1)^2 - \varepsilon(n)$.

3.4. We now consider a subset A of n vertices of Q_k $(n \geqslant 2,\ k \geqslant n-1)$ such that

$$D(A) \leqslant (n-1)^2 - \varepsilon(n). \tag{I$_1$}$$

We shall show that

(ii) *The subgraph G of Q_k induced by A is isomorphic to an n-star (when $n \neq 4$, $n \neq 8$) or to G_n (when $n \in \{4,..., 8\}$).*

This, together with **3.3**(i), will complete the proof of the theorem. We denote by m the number of edges of G and by Δ its maximum degree.

3.4.1. (ii) *is true when* $\Delta \leqslant 1$. Clearly, $D(A) \geqslant m + 2(n(n-1)/2 - m) = n(n-1) - m$. Then by (I_1), $n(n-1) - m \leqslant (n-1)^2$, and hence $m \geqslant n-1$. Since $n \geqslant 2$, $m \geqslant 1$ and we must have $\Delta = 1$. Then $2n - 2 \leqslant 2m \leqslant n$. It follows that $n = 2$, $m = 1$ and G is the 2-star.

3.4.2. (ii) *is true when* $\Delta = 2$. Let A_i be the set of vertices of A of color i, and let $|A_i| = a_i$ $(i = 0, 1)$. The distance between two vertices is even iff they have the same color. It easily follows that

$$D(A) \geqslant m + 3(a_0 a_1 - m) + 2\left(\frac{a_0(a_0-1)}{2} + \frac{a_1(a_1-1)}{2}\right),$$

or equivalently, $D(A) \geqslant 3a_0 a_1 + a_0(a_0 - 1) + a_1(a_1 - 1) - 2m$. By (I_1),

$D(A) \leqslant (a_0 + a_1 - 1)^2$. It follows that $a_0^2 + a_1^2 + 2a_0a_1 - 2(a_0 + a_1) + 1 \geqslant 3a_0a_1 + a_0^2 + a_1^2 - a_0 - a_1 - 2m$. This reduces to $2m \geqslant a_0a_1 + a_0 + a_1 - 1$.

We may assume without loss of generality that $a_0 \leqslant a_1$. Since $\Delta = 2$, we have $m \leqslant 2a_0$. Using these three inequalities we may write

$$4a_0 \geqslant 2m \geqslant a_0a_1 + a_0 + a_1 - 1 \geqslant a_0^2 + 2a_0 - 1. \qquad (\mathrm{I}_2)$$

Hence $a_0^2 - 2a_0 - 1 \leqslant 0$. It follows that $a_0 \leqslant 2$.

If $a_0 = 2$, (I_2) gives $8 \geqslant 2m \geqslant 3a_1 + 1$. Since $a_1 \geqslant a_0 = 2$, we must have $a_1 = 2$, and hence $n = 4$, $m = 4$. Then G is isomorphic to the square G_4.

If $a_0 < 2$, since $\Delta = 2$, we must have $a_0 = 1$. Then (I_2) gives $4 \geqslant 2m \geqslant 2a_1$. Hence $a_1 \leqslant 2$, and we must have equality since $\Delta = 2$. It follows that $n = 3$, $m = 2$, and G is the 3-star.

3.4.3. (ii) *is true when* $n \leqslant 4$. If $\Delta \leqslant 2$, the result is proved in 3.4.1, 3.4.2. Otherwise $3 \leqslant \Delta \leqslant n - 1$ implies that $n = 4$, $\Delta = 3$, and G is the 4-star. But then $D(A) = 9$, and this is a contradiction with (I_1), so that this case cannot occur.

3.4.4. In the sequel we assume that $n \geqslant 5$, $\Delta \geqslant 3$, and we choose a vertex x of degree Δ in G. We denote by B the set of vertices of A equal or adjacent to x, by C the set $A - B$, and by $D(B, C)$ the sum $\sum_{y \in B, z \in C} d(y, z)$. Clearly the subgraph of Q_k induced by B is a $(\Delta + 1)$-star, so that $D(B) = \Delta^2$. Moreover $D(A) = D(B) + D(C) + D(B, C)$, and hence,

$$D(A) = \Delta^2 + D(C) + D(B, C). \qquad (\mathrm{I}_3)$$

3.4.5. Consider a vertex z of C. By the definition of C, $d(z, x) \geqslant 2$.

3.4.5.1. If $d(z, x) = 2$, denote by B_z the set of vertices of B adjacent to z. Then x and z have the same color, and hence the vertices of $B - \{x\}$ are at odd distance from z. It follows that the vertices of $(B - \{x\}) - B_z$ are at distance 3 from z. Hence,

$$\sum_{y \in B} d(z, y) = d(z, x) + |B_z| + 3|(B - \{x\}) - B_z| = 3\Delta + 2 - 2\,|B_z|.$$

By the lemma, $|B_z| \leqslant 2$, so that $\sum_{y \in B} d(z, y) \geqslant 3\Delta - 2$.

3.4.5.2. If $d(z, x) = 3$, denote by B'_z the set of vertices of B at distance 2 from z. Then x and z have different colors, and hence the vertices of $B - \{x\}$ are at even distance from z. It follows that the vertices of $(B - \{x\}) - B'_z$ are at distance 4 from z. Hence,

$$\sum_{y \in B} d(z, y) = d(z, x) + 2\,|B'_z| + 4\,|(B - \{x\}) - B'_z| = 4\Delta + 3 - 2\,|B'_z|.$$

By the lemma, $|B'_z| \leqslant 3$, so that $\sum_{y \in B} d(z, y) \geqslant 4\Delta - 3$.

3.4.5.3. If $d(z, x) \geqslant 4$, every vertex of $B - \{x\}$ is at distance at least 3 from z. Hence

$$\sum_{y \in B} d(z, y) \geqslant d(z, x) + 3\,|B - \{x\}| \geqslant 3\Delta + 4.$$

3.4.5.4. The case $\Delta = 3$ will be of special interest. We note that if $d(z, x) = 2$ and $|B_z| = 2$, $\sum_{y \in B} d(z, y) = 7$ (see Sect. 3.4.5.1). In any other case, $\sum_{y \in B} d(z, y) \geqslant 9$.

3.4.6. (ii) *is true when* $n \leqslant 6$. For $n = 5$, we must have $\Delta \leqslant 4$. If $\Delta = 4$, G is the 5-star and we are done. Otherwise $\Delta = 3$. Then $|C| = 1$, so that $D(C) = 0$. By (I_3), $D(A) = 9 + D(B, C)$. By (I_1), $D(A) \leqslant 16$. It follows that $D(B, C) \leqslant 7$. By 3.4.5.4, this is true if and only if the unique vertex of C is adjacent to exactly two vertices of B. Then G is isomorphic to G_5.

For $n = 6$, we must have $\Delta \leqslant 5$. If $\Delta = 5$, G is the 6-star and we are done. If $\Delta = 4$, then $|C| = 1$, so that $D(C) = 0$. By (I_3), $D(A) = 16 + D(B, C)$. By (I_1), $D(A) \leqslant 25$. It follows that $D(B, C) \leqslant 9$. By 3.4.5.1, 3.4.5.2, and 3.4.5.3, $D(B, C) \geqslant 10$ and this case is impossible. Finally if $\Delta = 3$, then $|C| = 2$. By (I_3), $D(A) = 9 + D(C) + D(B, C)$. By (I_1), $D(A) \leqslant 25$. It follows that $D(C) + D(B, C) \leqslant 16$. Since $D(C) \geqslant 1$, we must have $D(B, C) \leqslant 15$.

By 3.4.5.4, this is true if and only if each of the two vertices of C is adjacent to exactly two vertices of B. By the lemma, the two vertices of C cannot be adjacent to the same pair of vertices of B. We conclude that G is isomorphic to G_6.

3.4.7. (ii) *is true for every* n. We prove this by induction on n. If $C = \varnothing$, G is an n-star and the result is proved (the 4-star and 8-star are excluded because of (I_1). Hence we may assume that $|C| \geqslant 1$. By 3.4.1 and 3.4.2, we also may assume that $\Delta \geqslant 3$. The initialization of the induction is given by 3.4.6. The induction hypothesis implies that $D(C) \geqslant (|C| - 1)^2 - \varepsilon(|C|)$ when $|C| \geqslant 2$. This remains true when $|C| = 1$ (we shall define $\varepsilon(1)$ as equal to zero). Hence we may write

$$D(C) \geqslant (n - \Delta - 2)^2 - \varepsilon(n - \Delta - 1) \geqslant (n - \Delta - 2)^2 - 1. \qquad (\mathrm{I}_4)$$

Since $\Delta \geqslant 3$, it follows from 3.4.5.1, 3.4.5.2, and 3.4.5.3 that for every vertex z in C, $\Sigma_{y \in B} d(z, y) \geqslant 3\Delta - 2$. Hence

$$D(B, C) \geqslant (n - \Delta - 1)(3\Delta - 2) \qquad (\mathrm{I}_5)$$

Then (I_3) together with (I_4) and (I_5) gives $D(A) \geqslant \Delta^2 + (n - \Delta - 2)^2 - 1 + (n - \Delta - 1)(3\Delta - 2)$, that is, $D(A) \geqslant \Delta^2 + (n - \Delta - 1)((n - \Delta - 3) + (3\Delta - 2))$.

It follows that $(n-1)^2-D(A)\leqslant(n-1)^2-\Delta^2-(n-\Delta-1)(n+2\Delta-5)$, that is, $(n-1)^2-D(A)\leqslant(n-\Delta-1)[(n-1+\Delta)-(n+2\Delta-5)]=(n-\Delta-1)(4-\Delta)$.

The left-hand side is non-negative by (I_1). We have assumed that $C\neq\varnothing$, so that $n-\Delta-1>0$. It follows that $\Delta\leqslant 4$.

If $\Delta=4$, clearly all the above inequalities must hold with equality. In particular, (I_5) holds with equality. This is true if and only if for every vertex z in C, $\Sigma_{y\in B}\, d(z,y)=3\Delta-2$. By 3.4.5, every vertex of C is adjacent to exactly two vertices of $B-\{x\}$. By the lemma, two vertices of C cannot be adjacent to the same pair of vertices of $B-\{x\}$. Since there are 6 distinct pairs of vertices of $B-\{x\}$, $|C|\leqslant 6$. Moreover, since (I_4) holds with equality, $\varepsilon(|C|)=1$. It follows from 3.4.6 that $|C|=4$ and the subgraph of Q_k induced by C is a square. Now a vertex of $B-\{x\}$ cannot be adjacent to two vertices of C (this would create a triangle or contradict the lemma). Hence the number of edges connecting B and C is at most 4, a contradiction. This case cannot occur.

If $\Delta=3$, we refine the above calculations. (I_4) gives

$$D(C)\geqslant(n-5)^2-\varepsilon(n-4). \qquad (I_4')$$

We denote by C_2 the set of vertices of C adjacent to exactly two vertices of B. Since no two vertices of C_2 are adjacent to the same pair of vertices of B, $|C_2|\leqslant 3$. It then follows from 3.4.5.4 that

$$D(B,C)\geqslant 7\,|C_2|+9(n-4-|C_2|)$$

and we may write

$$D(B,C)\geqslant 9n-36-2\,|C_2|\geqslant 9n-42. \qquad (I_5')$$

Then (I_3) together with (I_4') and (I_5') gives

$$\begin{aligned}D(A)&\geqslant 9+9n-42+(n-5)^2-\varepsilon(n-4)\\&=n^2-n-8-\varepsilon(n-4).\end{aligned}$$

It follows that $(n-1)^2-D(A)\leqslant(n^2-2n+1)-[n^2-n-8-\varepsilon(n-4)]$, that is, $(n-1)^2-D(A)\leqslant 9+\varepsilon(n-4)-n\leqslant 10-n$.

The left-hand side is non-negative by (I_1). Hence $n\leqslant 10$. The case $n=10$ is excluded because $\varepsilon(6)=0$.

For $n=9$, since $\varepsilon(5)=0$, all inequalities must hold with equality. The equality in (I_5') implies that $|C_2|=3$. Since $|C|=5$ and (I_4') holds with equality, it follows from 3.4.6 that the subgraph of Q_k induced by C is isomorphic to G_5 or to the 5-star. In both cases some vertex of C must be of degree greater than 3 in G, a contradiction.

For $n=8$, (I'_4) becomes $D(C) \geqslant 9-\varepsilon(4)=8$ and (I'_5) gives $D(B, C) \geqslant 36-2|C_2|$. Using ($I_3$) we obtain $D(A) \geqslant 9+8+36-2|C_2|=53-2|C_2|$. On the other hand, ($I_1$) gives $D(A) \leqslant 48$. Hence $2|C_2| \geqslant 53-48=5$, so that $|C_2|=3$. Now $D(B, C) \geqslant 30$. Since $9+D(C)+D(B, C)=D(A) \leqslant 48$, we must have $D(C) \leqslant 9$.

Moreover distinct vertices of C_2 are adjacent to distinct pairs of verices of $B-\{x\}$, so that the edges with one end in C_2 and the other in $B-\{x\}$, form a cycle of length 6. Then no two vertices of C_2 are adjacent (otherwise we would obtain a triangle). Hence $D(C_2) \geqslant 6$. Then $D(C) \leqslant 9$ implies that the vertex of $C-C_2$ is adjacent to all vertices of C_2. It follows that G is isomorphic to the 3-dimensional cube G_8.

By 3.4.6, the only remaining case is $n=7$. Then (I'_4) becomes $D(C) \geqslant 4-\varepsilon(3)=4$ and (I'_5) gives $D(B, C) \geqslant 27-2|C_2|$. Using ($I_3$), we obtain $D(A) \geqslant 9+4+27-2|C_2|=40-2|C_2|$. On the other hand, ($I_1$) gives $D(A) \leqslant 36$. Hence $2|C_2| \geqslant 40-36$, so that $|C_2| \geqslant 2$. If $|C_2|=3$, every vertex of C is adjacent to two vertices of $B-\{x\}$, and no two vertices of C are adjacent to the same pair of vertices of $B-\{x\}$. Then G is clearly isomorphic to G_7. Otherwise, $|C_2|=2$. Then $D(B, C) \geqslant 23$. Since $36 \geqslant D(A)=9+D(C)+D(B, C)$, we must have $D(C) \leqslant 4$, so that $D(C)=4$. By (ii) (applied to C) the subgraph of Q_k induced by C is a 3-star. Since $\Delta=3$, the vertex of degree two of this 3-star is the vertex of $C-C_2$. Then again G is easily seen to be isomorphic to G_7. This completes the proof of Theorem 1.

4. Consequences

4.1. Using the correspondence described in Section 2.2, it is easy to reformulate Theorem 1 in terms of cocycle covers. For the sake of simplicity, we denote the vertex-set of K_n by $\{1,..., n\}$. Two cocycle covers $C=(\omega_{K_n}(S_i),\ i=1,..., k)$ and $C'=(\omega_{K_n}(S'_1),\ i=1,..., k)$ will be said to be isomorphic when there exists a bijection σ of $\{1,..., n\}$ such that $\forall i \in \{1,..., k\}$, $\omega_{K_n}(S_i)=\omega_{K_n}(\sigma(S'_i))$. Then we obtain

Theorem 2. *For every $n \geqslant 2$, $l(K_n)=(n-1)^2-\varepsilon(n)$, where $\varepsilon(n)=1$ if $n=4$ or $n=8$, $\varepsilon(n)=0$ otherwise. Moreover a cocycle cover of K_n (consisting of non-empty cocycles) has length $l(K_n)$ if and only if it is isomorphic to one of the following cocycle covers*:

—$(\omega_{K_n}(\{1\}).\ i=1,..., n-1)$ *for all* $n \geqslant 2$, $n \neq 4$, $n \neq 8$

—$(\omega_{K_n}(\{1, 2\}),\ \omega_{K_n}(\{1, 3\}))$ *for* $n=4$

—$(\omega_{K_n}(S_i),\ i=1, 2, 3)$ *for* $n=5, 6, 7, 8$, *where* $S_1=\{1, 2, 3, 4\}$, $S_2=\{1, 2, 7, 8\} \cap \{1,..., n\}$, $S_3=\{1, 4, 5, 8\} \cap \{1,..., n\}$.

The proof is left to the reader.

4.2. PROPOSITION 2. *Let $G=(V, E)$ be a (loopless) n-colorable graph. Then the minimum length of a cocycle cover of G is at most.*

- $\frac{4}{3}|E|$ *for* $n=4$
- $\frac{12}{7}|E|$ *for* $n=8$
- $(2-2/n)|E|$ *for every* $n \geqslant 2$, $n \neq 4$, $n \neq 8$.

Moreover equality holds when G is the complete graph K_n.

Proof. The fact that equality holds when G is the complete graph K_n follows from Theorem 2.

Consider a coloring of G with n colors ($n \geqslant 2$), and denote by V_i ($i=1,..., n$) the set of vertices which have received the ith color. We assume, without loss of generality that $|\omega_G(V_i)| \leqslant |\omega_G(V_n)|$ for all $i=1,..., n$.

Clearly $C=(\omega_G(V_i),\ i=1,..., n-1)$ is a cocycle cover of G, and $\sum_{i=1}^{n} |\omega_G(V_i)| = 2|E|$. It follows that C has length $2|E| - |\omega_G(V_n)|$, and that $|\omega_G(V_n)| \geqslant 2|E|/n$. Hence the length of C is at most $(2-2/n)|E|$. This settles the general case.

For $n=4$, we observe that every edge of G appears in exactly two of the three cocycles $\omega_G(V_1 \cup V_2)$, $\omega_G(V_1 \cup V_3)$, $\omega_G(V_1 \cup V_4)$. By deleting the largest one, we obtain a cocycle cover of G of length at most $\frac{4}{3}|E|$.

For $n=8$, let Q be the set of 4-element subsets of $\{1,..., 8\}$. Let T be the set of 3-element subsets $\{q_1, q_2, q_3\}$ of Q with the following property: for every pair $\{i, j\}$ of distinct elements of $\{1,..., 8\}$, there exists $k \in \{1, 2, 3\}$ such that q_k contains exactly one of the elements i, j. Then for every $t=\{q_1, q_2, q_3\} \in T$, $(\omega_G(\bigcup_{i \in q_j} V_i), j=1, 2, 3)$ is a cocycle cover of G, which we denote by C_t.

For $q \in Q$, let $k(q)$ be the number of elements of T which contain q. It is easy to see that $k(q)$ is independent of q. We denote the common value of all numbers $k(q)$ ($q \in Q$) by k. Then $\sum_{t \in T} l(C_t) = k \sum_{q \in Q} |\omega_G(\bigcup_{i \in q} V_i)|$.

Moreover for every pair i, j of distinct elements of $\{1,..., 8\}$, there exist exactly 40 elements q of Q such that q contains exactly one of the elements i, j. Equivalently, for every edge e of G there exists exactly 40 elements q of Q such that $e \in \omega_G(\bigcup_{i \in q} V_i)$. Hence $\sum_{q \in Q} |\omega_G(\bigcup_{i \in q} V_i)| = 40|E|$. It follows that the average length of a cocycle cover of the form C_t ($t \in T$) is $40k|E|/|T|$.

Now by counting in two ways the number of pairs (q, t) such that $q \in Q$, $t \in T$, and $q \in t$, we obtain $k|Q| = 3|T|$. Since $|Q| = C_8^4 = 70$, we have that $k/|T| = \frac{3}{70}$; hence the average length of a cocycle cover of the form C_t ($t \in T$) is $\frac{3}{70} \times 40|E| = \frac{12}{7}|E|$. We conclude that some cocycle cover of G has length at most $\frac{12}{7}|E|$. This completes the proof.

An immediate consequence of this result is the following:

4.3. PROPOSITION 3. *For every $\alpha < 2$, there exists a loopless graph $G = (V, E)$, each cocycle cover of which has length greater than $\alpha\,|E|$.*

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